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Minimal G-Functions

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ABSTRACT

The concept of a G-function has been introduced by Nowosad and Hoffman; it gives an appropriate setting for many generalizations of the Gerschgorin Circle Theorem. In this paper we establish several equivalent conditions for a minimal continuous G-function and for a minimal G-function, and give characterizations of such minimal functions. We show that a convolution of two minimal G-functions is seldom minimal. Finally, we establish new results concerning the patterns of dependence of G-functions.

1. INTRODUCTION

The concept of a G-function has been introduced by Nowosad and Hoffman; it gives an appropriate setting for many generalizations of the Gerschgorin Circle Theorem. In this paper we establish several equivalent conditions for a minimal continuous G-function and for a minimal G-function, and give characterizations of such minimal functions. We show that a convolution of two minimal G-functions is seldom minimal. Finally, we establish new results concerning the patterns of dependence of G-functions.

2. NOTATION AND PRELIMINARY RESULTS

Let $\mathbf{C}^{n,n}$ denote the set of all $n \times n$ complex matrices. Let $\mathscr{P}_n$, $n \geqslant 2$, be the collection of all functions $f = (f_1, \ldots, f_n)$ such that for each $i =$

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$1, 2, \ldots, n$, $f_i: \mathbf{C}^{n,n} \to \mathbf{R}_+$, i.e., $\infty > f_i(A) \geqslant 0$ for any $A \in \mathbf{C}^{n,n}$, and f_i depends only on the moduli of the off-diagonal entries of the matrices, i.e., if $B = (b_{i,j})$ and $A = (a_{i,j})$ are in $\mathbf{C}^{n,n}$ with $|b_{i,j}| = |a_{i,j}|$ for all $i, j = 1, 2, \ldots, n$, $i \neq j$, then $f_i(B) = f_i(A)$. We begin with (cf. Hoffman [2], Hoffman and Varga [3], and Nowosad [5, 6])

DEFINITION 1. We say $f \in \mathscr{P}_n$ is a *G-function* if, for each $A = (a_{i,j}) \in \mathbf{C}^{n,n}$ satisfying

$$|a_{i,i}| > f_i(A), \qquad i = 1, 2, \ldots, n, \tag{2.1}$$

A is nonsingular.

Equivalently, $f \in \mathscr{P}_n$ is a G-function if, for every $A = (a_{i,j}) \in \mathbf{C}^{n,n}$, every eigenvalue of A lies in the union of the n disks

$$\Delta_k = \{z \in \mathbf{C}: |z - a_{k,k}| \leqslant f_k(A)\}, \qquad k = 1, 2, \ldots, n. \tag{2.2}$$

We will denote by $\mathscr{G}_n$ the set of G-functions in $\mathscr{P}_n$.

As examples, if

$$r_i(A) \equiv \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|, \qquad c_i(A) \equiv \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{j,i}|, \qquad i = 1, 2, \ldots, n, \tag{2.3}$$

then $r = (r_1, \ldots, r_n)$ and $c = (c_1, \ldots, c_n)$ are G-functions. More generally, if $x = (x_1, \ldots, x_n)^T$ is any column vector in $\mathbf{C}^n$ with positive components, written $x > 0$, and

$$r_i^x(A) \equiv \frac{1}{x_i} \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}| x_j, \quad c_i^x(A) \equiv \frac{1}{x_i} \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{j,i}| x_j, \quad i = 1, 2, \ldots, n, \tag{2.4}$$

then $r^x = (r_1^x, \ldots, r_n^x)$ and $c^x = (c_1^x, \ldots, c_n^x)$ are G-functions.

The study of G-functions is closely related to the study of M-matrices[1], as is shown by the following proposition, which follows easily from the initial work in this area by Ostrowski [7], as well as a result of Fan [1]. For notation, if $f \in \mathscr{P}_n$ and if $A = (a_{i,j}) \in \mathbf{C}^{n,n}$, then $\mathscr{M}^f(A) = (\alpha_{i,j}) \in \mathbf{C}^{n,n}$ is the matrix whose elements are defined by

$$\alpha_{i,j} = -|a_{i,j}| \quad \text{for all} \quad i \neq j, \qquad \alpha_{i,i} = f_i(A), \qquad i, j = 1, 2, \ldots, n. \tag{2.5}$$

[1] $B = (b_{i,j}) \in \mathbf{R}^{k,k}$, $k \geqslant 1$, is a (possibly singular) *M-matrix* if and only if $b_{i,j} \leqslant 0$ for all $i \neq j$, and for any $d = (d_1, \ldots, d_k)^T \in \mathbf{C}^k$ with $d > 0$, $B + \operatorname{diag}(d_1, \ldots, d_k)$ is nonsingular. If B is an irreducible M-matrix, then $B + \operatorname{diag}(d_1, \ldots, d_k)$ is nonsingular for any $d \geqslant 0$ with $d \neq 0$.

PROPOSITION 1. *Let $f \in \mathscr{P}_n$. Then $f \in \mathscr{G}_n$ if and only if $\mathscr{M}^f(A)$ is a (possibly singular) M-matrix for every $A \in \mathbf{C}^{n,n}$. Thus, if $f \in \mathscr{G}_n$ and if $A \in \mathbf{C}^{n,n}$ is irreducible, there exists an $x \in \mathbf{C}^n$ with $x > 0$ (depending on A) such that*

$$f_i(A) \geqslant r_i^x(A), \qquad i = 1, 2, \ldots, n; \tag{2.6}$$

when $\mathscr{M}^f(A)$ is singular, equality holds for all $i = 1, 2, \ldots, n$, i.e.,

$$f_i(A) = r_i^x(A), \qquad i = 1, 2, \ldots, n. \tag{2.7}$$

It follows from Eq. (2.6) that if $f \in \mathscr{G}_n$ and $A \in \mathbf{C}^{n,n}$ is irreducible, then $f_i(A) > 0$, $i = 1, 2, \ldots, n$. Note also that the vector x of Eq. (2.7) is the unique (up to scalar factors) eigenvector associated with the zero eigenvalue of the irreducible matrix $\mathscr{M}^f(A)$.

We shall say that $f \in \mathscr{P}_n$ is *continuous* if, for each $i = 1, 2, \ldots, n$, f_i is continuous on all of $\mathbf{C}^{n,n}$. The set of continuous $f \in \mathscr{P}_n$ and $\mathscr{G}_n$ will be denoted by $\mathscr{P}_n{}^c$ and $\mathscr{G}_n{}^c$, respectively. Since the set of irreducible $n \times n$ matrices is dense in $\mathbf{C}^{n,n}$, it is clear that if $f \in \mathscr{P}_n$ is continuous, it is completely determined by its action on the irreducible matrices.

PROPOSITION 2. *Let $f \in \mathscr{P}_n{}^c$. Suppose that for every irreducible $A \in \mathbf{C}^{n,n}$ which satisfies Eq.* (2.1), *A is nonsingular. Then, $f \in \mathscr{G}_n{}^c$.*

Proof. Let A be any reducible matrix in $\mathbf{C}^{n,n}$ which satisfies Eq. (2.1). We must show that $A = (a_{i,j})$ is nonsingular. For $\varepsilon > 0$, define $A(\varepsilon) = [a_{i,j}(\varepsilon)] \in \mathbf{C}^{n,n}$ by

$$a_{i,j}(\varepsilon) = \begin{cases} -|a_{i,j}| & \text{if} \quad i \neq j \quad \text{and} \quad a_{i,j} \neq 0; \\ -\varepsilon & \text{if} \quad i \neq j \quad \text{and} \quad a_{i,j} = 0; \\ |a_{i,i}| & \text{if} \quad i = j. \end{cases} \tag{2.8}$$

For $\varepsilon > 0$ sufficiently small, it is clear from (2.1) and the continuity of f, that $A(\varepsilon)$ satisfies (2.1), and is irreducible as well. Thus, by hypothesis, $A(\varepsilon)$ is nonsingular, and $A(\varepsilon)$ is evidently a nonsingular M-matrix. But, because the entries $a_{i,j}$ of A satisfy

$$|a_{i,j}| \leqslant |a_{i,j}(\varepsilon)| \quad \text{for all} \quad i \neq j, \qquad |a_{i,i}| = a_{i,i}(\varepsilon), \qquad i,j = 1, 2, \ldots, n, \tag{2.9}$$

and because $A(\varepsilon)$ is a nonsingular M-matrix, then it follows (cf. Ostrowski [7]) that $|\det A| \geqslant \det A(\varepsilon) > 0$, i.e., A is nonsingular. Q.E.D.

3. THE CONVEX STRUCTURE OF $\mathcal{G}_n$ AND $\mathcal{G}_n^c$

We first define a partial order on $\mathcal{P}_n$. If f and g are in $\mathcal{P}_n$, we write

$$f \geqslant g \quad \text{if} \quad f_i(A) \geqslant g_i(A), \qquad i = 1, 2, \ldots, n, \quad \text{all} \quad A \in \mathbf{C}^{n,n}. \tag{3.1}$$

It is clear from Proposition 1 that if $f \in \mathcal{P}_n$ and $g \in \mathcal{G}_n$, with $f \geqslant g$, then also $f \in \mathcal{G}_n$.

Next we state a theorem of Hoffman [2]; we shall in Sec. 4 prove a slight extension (Theorem 3), and use our proof to obtain other results.

THEOREM 1. *If f and g are in $\mathcal{G}_n$, and $0 < \alpha < 1$, then h, defined by*

$$h_i(A) = f_i^{\alpha}(A) g_i^{1-\alpha}(A), \qquad i = 1, 2, \ldots, n, \quad \textit{all} \quad A \in \mathbf{C}^{n,n}, \tag{3.2}$$

is also in $\mathcal{G}_n$.

We shall call the G-function h, defined by Eq. (3.2), the *α-convolution* of f and g. As has been noted by Hoffman, it follows from Theorem 1 that $\mathcal{G}_n$ and $\mathcal{G}_n^c$ are *convex sets*. To see this, given f and g in $\mathcal{G}_n$ and $0 < \alpha < 1$, define $k = (k_1, \ldots, k_n) \in \mathcal{P}_n$ by

$$k_i(A) = \alpha f_i(A) + (1 - \alpha) g_i(A), \qquad i = 1, 2, \ldots, n, \quad \text{all} \quad A \in \mathbf{C}^{n,n}. \tag{3.3}$$

By the generalized arithmetic-geometric mean inequality, $k \geqslant h$. Since $h \in \mathcal{G}_n$, we have $k \in \mathcal{G}_n$, i.e., $\mathcal{G}_n$ is convex. Obviously if f and g are in $\mathcal{G}_n^c$, so are h and k, and $\mathcal{G}_n^c$ is also convex.

We next define

$$\mathcal{L}_n = \{\ln f = (\ln f_1, \ldots, \ln f_n)\colon \ f = (f_1, \ldots, f_n) \in \mathcal{G}_n\}, \tag{3.4}$$

and $\mathcal{L}_n^c$ analogously; note that the $\ln f_i$ may assume the value $-\infty$. Hoffman's result may be restated as: $\mathcal{L}_n$ is convex (and, hence, so is $\mathcal{L}_n^c$).

Suppose f and g are in $\mathcal{G}_n$, with $f \geqslant g$. Then, for any $A \in \mathbf{C}^{n,n}$, f determines larger eigenvalue inclusion regions [cf. Eq. (2.2)] than g, and is thus, in a sense, uninteresting. We may cull out such uninteresting G-functions with the following

DEFINITION 2. Let $f \in \mathcal{G}_n$.

(i) f is *minimal in $\mathcal{G}_n$* (or *minimal*) if, for every $g \in \mathcal{G}_n$ for which $g \leqslant f$, we have $g = f$;

(ii) if $f \in \mathcal{G}_n^c$, i.e., if f is also continuous, then f is *minimal in $\mathcal{G}_n^c$* (*or minimal continuous*) if, for every $g \in \mathcal{G}_n^c$ for which $g \leqslant f$, we have $g = f$.

The minimal elements of the convex sets $\mathscr{G}_n$ and $\mathscr{G}_n^c$ are in fact the extreme points of $\mathscr{G}_n$ and $\mathscr{G}_n^c$. [An *extreme point* f of a convex set C is such that if $f = \alpha g + (1 - \alpha)h$, where $0 < \alpha < 1$ and $g, h \in C$, then $f = g = h$ (cf. [9, p. 162]).] Suppose $f \in \mathscr{G}_n$ is not minimal; then there exists $g \in \mathscr{G}_n$, $g \leqslant f$, $g \neq f$. If we define $h = 2f - g$, then $h \geqslant f$, $h \in \mathscr{G}_n$, and $f = \frac{1}{2}g + \frac{1}{2}h$ is not extreme in $\mathscr{G}_n$. On the other hand, if $f \in \mathscr{G}_n$ is not extreme in $\mathscr{G}_n$, then $f = \alpha g + (1 - \alpha)h$, where $0 < \alpha < 1$, $g, h \in \mathscr{G}_n$, and $g \neq h$. Since $g \neq h$, there is an $A \in \mathbf{C}^{n,n}$ for which, for some i, $g_i(A) \neq h_i(A)$. For this A and this i, we have

$$g_i^{\alpha}(A)h_i^{1-\alpha}(A) < \alpha g_i(A) + (1 - \alpha)h_i(A) = f_i(A). \tag{3.5}$$

Thus, we have $g^{\alpha}h^{1-\alpha} \leqslant \alpha g + (1 - \alpha)h = f$, and $g^{\alpha}h^{1-\alpha} \neq f$, so that f is not minimal in $\mathscr{G}_n$. The same arguments apply to $\mathscr{G}_n^c$.

4. MINIMAL CONTINUOUS G-FUNCTIONS

It follows from Eq. (2.6) of Proposition 1 that, for a G-function f which is minimal in $\mathscr{G}_n$, we must have $f(A) = r^x(A)$ for each irreducible $A \in \mathbf{C}^{n,n}$, where the vector $x > 0$ depends on A. We will show in this section that this property holds for any f which is minimal in $\mathscr{G}_n^c$, and in fact is equivalent to minimality in $\mathscr{G}_n^c$. In the succeeding section, we find a generalization of this property, to include reducible matrices, which is equivalent to minimality in $\mathscr{G}_n$.

Let $\mathscr{I}_n^c$, $n \geqslant 2$, denote the collection of all functions $g = (g_1, \ldots, g_n)$, where, for each $i = 1, 2, \ldots, n$, g_i is defined, positive, and continuous on the set of irreducible matrices in $\mathbf{C}^{n,n}$, and depends only on the moduli of offdiagonal entries. For $g \in \mathscr{I}_n^c$, we define $r^g = (r_1^g, \ldots, r_n^g) \in \mathscr{I}_n^c$ by

$$r_i^g(A) = \frac{1}{g_i(A)} \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{i,j}| g_j(A), \qquad i = 1, 2, \ldots, n, \tag{4.1}$$

for each irreducible $A \in \mathbf{C}^{n,n}$.

THEOREM 2. *Let $f \in \mathscr{G}_n^c$. Then the following are equivalent:*

(i) *f is minimal in $\mathscr{G}_n^c$;*

(ii) *f is an extreme point of the convex set $\mathscr{G}_n^c$;*

(ii′) $(n > 2)$ $\ln f$ *is an extreme point of the convex set $\mathscr{L}_n^c$;*

(iii) *for every $A \in \mathbf{C}^{n,n}$, the matrix $\mathscr{M}^f(A)$ is singular;*

(iii′) *for every $A \in \mathbf{C}^{n,n}$, there exists a $B \in \mathbf{C}^{n,n}$ with*

$$|b_{i,j}| = |a_{i,j}| \quad \textit{for all} \quad i \neq j, \qquad |b_{i,i}| = f_i(A), \qquad i, j = 1, 2, \ldots, n, \tag{4.2}$$

for which B *singular*;

(iv) *for every irreducible* $A \in \mathbf{C}^{n,n}$, *there exists an* $x \in \mathbf{C}^n$ *with* $x > 0$ (*depending on* A) *for which*

$$f_i(A) = r_i^x(A), \qquad i = 1, 2, \ldots, n; \tag{4.3}$$

(iv′) *there exists a* $g \in \mathscr{I}_n^c$ *such that, for every irreducible* $A \in \mathbf{C}^{n,n}$,

$$f_i(A) = r_i^g(A), \qquad i = 1, 2, \ldots, n. \tag{4.4}$$

Remark. Because f is continuous, conditions (iii) and (iii′) could be restricted to matrices $A \in \mathbf{C}^{n,n}$ which are irreducible.

Proof. That (i) and (ii) are equivalent has already been proved. That (i) implies (ii′) for $n > 2$ will be proved later in this section. To prove that (ii′) implies (i) for all $n \geqslant 2$, assume that f is not minimal in $\mathscr{G}_n^c$. Thus, there exist a $g \in \mathscr{G}_n^c$ with $g \leqslant f$ and a matrix $\tilde{A}$ such that $g_j(\tilde{A}) < f_j(\tilde{A})$ for some j, $1 \leqslant j \leqslant n$. By continuity, we may assume that $\tilde{A}$ is *irreducible.* Next, regarding $\tilde{A}$ as a point in the nonnegative hyperoctant of $\mathbf{R}_+^{n(n-1)}$, it is clear again from continuity that we can redefine $g \in \mathscr{G}_n^c$ so that $g \leqslant f$, but with $g \equiv f$, except on an ε-neighborhood of $\tilde{A}$. For ε sufficiently small, this ε-neighborhood of $\tilde{A}$ contains only irreducible matrices. Thus, f and g differ on the irreducible matrix $\tilde{A}$, but are identical on any reducible $A \in \mathbf{C}^{n,n}$. We can now define $h \in \mathscr{P}_n^c$ by

$$h_i(A) = \begin{cases} f_i(A) = g_i(A), & i = 1, 2, \ldots, n, \quad \text{if} \quad A \in \mathbf{C}^{n,n} \quad \text{is reducible}, \\ f_i^2(A) g_i^{-1}(A), & i = 1, 2, \ldots, n, \quad \text{if} \quad A \in \mathbf{C}^{n,n} \quad \text{is irreducible}. \end{cases} \tag{4.5}$$

For irreducible $A \in \mathbf{C}^{n,n}$, it follows from Proposition 1 that, for all $i = 1, 2, \ldots, n$, $g_i(A) > 0$, $h_i(A)$ is defined, and $h_i(A) = f_i^2(A) g_i^{-1}(A) \geqslant f_i(A)$. Thus, actually $h \in \mathscr{G}_n^c$. Now it is easy to see that $\ln f = \frac{1}{2} \ln g + \frac{1}{2} \ln h$, so that $\ln f$ is not extreme in $\mathscr{L}_n^c$.

That (iii) implies (iii′) is obvious. Conversely, since $f \in \mathscr{G}_n$, then from Proposition 1, $\mathscr{M}^f(A)$ is an M-matrix for any $A \in \mathbf{C}^{n,n}$. But, for all B satisfying Eq. (4.2), it follows (cf. [7]) that $|\det B| \geqslant \det \mathscr{M}^f(A) \geqslant 0$. Clearly, (iii′) implies (iii), and (iii) and (iii′) are thus equivalent.

That (iii) implies (iv) follows from Proposition 1. To show the converse, chose any irreducible $A \in \mathbf{C}^{n,n}$, and define $X = \operatorname{diag}(x_1, \ldots, x_n)$ for any $x > 0$ in $\mathbf{C}^n$. Assuming (iv), if $e = (1, 1, \ldots, 1)^T$, then Eq. (4.3) becomes $\mathscr{M}^f(A) X e = 0$, which implies that $\mathscr{M}^f(A) X$ and $\mathscr{M}^f(A)$ are singular. By our remark, this is sufficient to imply (iii).

We next show that (i) is equivalent to (iii). Suppose (i) does not hold, and that $g \in \mathscr{G}_n{}^c$ is such that $g \leqslant f$, $g \neq f$. There then exists an $A \in \mathbf{C}^{n,n}$ and an integer j with $1 \leqslant j \leqslant n$, for which

$$g_i(A) \leqslant f_i(A), \qquad i = 1, 2, \ldots, n, \quad \text{and} \quad g_j(A) < f_j(A). \tag{4.6}$$

Since f and g are both continuous, we may assume that A is irreducible. Since $\mathscr{M}^g(A)$ is, using Proposition 1, an irreducible M-matrix, the inequalities of Eq. (4.6) give us that $\mathscr{M}^f(A)$ is nonsingular, and (iii) does not hold, i.e., (iii) implies (i).

For any $f \in \mathscr{G}_n{}^c$, we can construct a $g \in \mathscr{G}_n{}^c$ with $g \leqslant f$ by defining $g_i(A)$ for every $A \in \mathbf{C}^{n,n}$ as

$$g_i(A) \equiv f_i(A) - \lambda(A), \qquad i = 1, 2, \ldots, n, \tag{4.7}$$

where $\lambda(A)$ is the minimal nonnegative real eigenvalue of the M-matrix $\mathscr{M}^f(A)$ (cf. [7]). Now, suppose (iii) does not hold, i.e., $\mathscr{M}^f(A)$ is nonsingular for some $A \in \mathbf{C}^{n,n}$. For this A, $\lambda(A) > 0$, and $g_i(A) < f_i(A)$ for all $i = 1, 2, \ldots, n$, i.e., f is not minimal in $\mathscr{G}_n{}^c$. Hence, (i) implies (iii).

Suppose that (iv) holds. The vector $x > 0$ of Eq. (4.3) is, as we noted after Proposition 1, the unique (up to scalar multiples) eigenvector for the null eigenvalue of $\mathscr{M}^f(A)$. It can be shown that, with proper normalization (e.g., choose $x_1 \equiv 1$ for all irreducible $A \in \mathbf{C}^{n,n}$), the vector $x \equiv g(A)$, defined for all irreducible $A \in \mathbf{C}^{n,n}$, depends continuously on the moduli of the off-diagonal entries of A. Thus, $g \in \mathscr{I}_n{}^c$, r^g is defined by Eq. (4.1), and, for all irreducible $A \in \mathbf{C}^{n,n}$, Eq. (4.3) becomes Eq. (4.4), i.e., (iv′) holds. That (iv′) implies (iv) is obvious. Q.E.D.

In general, for $g \in \mathscr{I}_n{}^c$, we cannot extend either g or r^g to all of $\mathbf{C}^{n,n}$. We shall discuss this further in Sec. 6. Note that in order to show that $f \in \mathscr{G}_n{}^c$ is *not* minimal in $\mathscr{G}_n{}^c$ it is sufficient by Theorem 2 to show that $\mathscr{M}^f(A)$ is nonsingular for some $A \in \mathbf{C}^{n,n}$.

COROLLARY. *For any $x \in \mathbf{C}^n$ with $x > 0$, r^x and c^x are minimal continuous G-functions.*

Another general example of a minimal continuous G-function is as follows. Given any $A = (a_{i,j}) \in \mathbf{C}^{n,n}$, let $\rho(A)$ be the maximal eigenvalue of the $n \times n$ nonnegative matrix $\mathscr{P}(A) = (|a_{i,j}| - \delta_{i,j}|a_{i,i}|)$, and define $f(A) = [f_1(A), \ldots, f_n(A)]$ by $f_i(A) \equiv \rho(A)$ for all $i = 1, 2, \ldots, n$. It is seen from (iii) of Theorem 2 that f is minimal in $\mathscr{G}_n{}^c$, and the associated $g(A) = [g_1(A), \ldots, g_n(A)]$ from Eq. (4.4) is, for any irreducible $A \in \mathbf{C}^{n,n}$, just a (normalized) positive eigenvector of $\mathscr{P}(A)$ corresponding to the eigenvalue $\rho(A)$. As our last example, it can be verified that $g = (g_1, g_2, g_3)$, defined on irreducible $A \in \mathbf{C}^{3,3}$ by

$$g_1(A) \equiv |a_{1,2}| + |a_{1,3}|, \qquad g_2(A) \equiv g_3(A) \equiv 1,$$

is an element of $\mathscr{I}_3{}^c$. In this case $r^g(A)$, for each irreducible $A \in \mathbf{C}^{3,3}$, is given by

$$(1, |a_{2,1}|\{|a_{1,2}| + |a_{1,3}|\} + |a_{2,3}|, |a_{3,1}|\{|a_{1,2}| + |a_{1,3}|\} + |a_{3,2}|),$$

and $f \in \mathscr{P}_3{}^c$, defined by the same rule, is clearly a minimal element of $\mathscr{G}_3{}^c$. In contrast with the G-functions r^x and c^x (with $x \in \mathbf{C}^n$, $x > 0$) and the above example, f, is *not* homogeneous [cf. Eq. (7.7)].

We give next the generalization of Theorem 1 promised in Section 3. For notation, if $A = (a_{i,j}) \in \mathbf{C}^{n,n}$, then $A^\alpha = (|a_{i,j}|^\alpha)$ for any $\alpha \geqslant 0$.

THEOREM 3. *If f and g are in $\mathscr{G}_n$, and $0 < \alpha, \beta < 1$, then h defined by*

$$h_i(A) = f_i^\alpha(A^{\beta/\alpha}) g_i^{1-\alpha}[A^{(1-\beta)/(1-\alpha)}], \quad i = 1, 2, \ldots, n, \quad \text{all} \quad A \in \mathbf{C}^{n,n}, \tag{4.8}$$

is also in $\mathscr{G}_n$.

Remark. We call h, defined by Eq. (4.8), the (α, β)-*convolution* of f and g. When $f = r$ and $g = c$ [cf. Eq. (2.3)], the theorem reduces to a result of Ostrowski [8]. When $\alpha = \beta$, we have Theorem 1. Our proof here is for $f, g \in \mathscr{G}_n{}^c$; the general proof can be similarly established using Corollary 2 to Theorem 6.

Proof. For $f, g \in \mathscr{G}_n{}^c$, it is clear that $h \in \mathscr{P}_n{}^c$. Thus, from Proposition 2, it is sufficient to prove that for any irreducible $A \in \mathbf{C}^{n,n}$ satisfying

$$|a_{ii}| > h_i(A), \qquad i = 1, 2, \ldots, n, \tag{4.9}$$

A is nonsingular. Choose an irreducible $A \in \mathbf{C}^{n,n}$; by Proposition 1, $\mathscr{M}^f(A^{\beta/\alpha})$ and $\mathscr{M}^g[A^{(1-\beta)/(1-\alpha)}]$ are irreducible M-matrices, so that there exist $x, y \in \mathbf{C}^n$, $x > 0$, $y > 0$, such that

$$f_i(A^{\beta/\alpha}) \geqslant \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|^{\beta/\alpha}(x_j/x_i); \qquad g_i[A^{(1-\beta)/(1-\alpha)}] \geqslant \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|^{(1-\beta)/(1-\alpha)}(y_j/y_i),$$
$$i = 1, \ldots, n. \tag{4.10}$$

As $0 < \alpha < 1$, the above inequalities and Hölder's inequality give

$$h_i(A) = f_i^{\alpha}(A^{\beta/\alpha})g_i^{1-\alpha}[A^{(1-\beta)/(1-\alpha)}]$$
$$\geqslant \left(\sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|^{\beta/\alpha} x_j/x_i\right)^{\alpha} \left(\sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|^{(1-\beta)/(1-\alpha)} y_j/y_i\right)^{1-\alpha}$$
$$\geqslant \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|(x_j^{\alpha}y_j^{1-\alpha}/x_i^{\alpha}y_i^{1-\alpha}) = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{i,j}|(z_j/z_i) = r_i^z(A), \tag{4.11}$$

where

$$z = (z_1, \ldots, z_n)^T \quad \text{is in } \mathbf{C}^n \text{ with } \quad z_i \equiv x_i^{\alpha}y_i^{1-\alpha} > 0 \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{4.12}$$

Thus, if A satisfies Eq. (4.9), then

$$|a_{i,i}| > h_i(A) = f_i^{\alpha}(A^{\beta/\alpha})g_i^{1-\alpha}[A^{(1-\beta)/(1-\alpha)}] \geqslant r_i^z(A), \qquad i = 1, 2, \ldots, n. \tag{4.13}$$

But since $r^z = (r_1^z, \ldots, r_n^z) \in \mathscr{G}_n$, A is evidently nonsingular. Q.E.D.

The results in this paper grew out of our attempts to answer the following question. If f and g are minimal in $\mathscr{G}_n^c$, and $0 < \alpha < 1$, is h, the α-convolution of f and g [cf. Eq. (3.2)], necessarily also minimal in $\mathscr{G}_n^c$? For $n = 2$ it is easy to verify that the answer is yes. The negative answer to this question for $n > 2$ is contained in our next theorem.

THEOREM 4. *For $n > 2$, no (α, β)-convolution with $0 < \alpha, \beta < 1$ of distinct (if $\alpha = \beta$) minimal elements of $\mathscr{G}_n^c$ is minimal in $\mathscr{G}_n^c$.*

Proof. Consider first the case when $n > 2$, $0 < \alpha = \beta < 1$, and f and g are distinct G-functions, minimal in $\mathscr{G}_n^c$. Since f and g are distinct, there exists an $A \in \mathbf{C}^{n,n}$ and an integer i, $1 \leqslant i \leqslant n$, for which $f_i(A) \neq g_i(A)$. By the continuity of f and g, we may assume that A has all nonzero offdiagonal entries (and is thus irreducible). Following now the proof of Theorem 3, the assumption that f and g are minimal continuous gives us, from (iv) of Theorem 2, that equality must hold throughout in Eq. (4.10) for all $i = 1, 2, \ldots, n$.

If h were minimal continuous, we would have, analogously, that equality holds throughout for all i in Eq. (4.11) in the application of Hölder's inequality. Hence, the vectors (for this case $\alpha = \beta$)

$$(|a_{i,j}|x_j/x_i)_{\substack{j=1\\ j\neq i}}^{n} \quad \text{and} \quad (|a_{i,j}|y_j/y_i)_{\substack{j=1\\ j\neq i}}^{n} \tag{4.14}$$

are proportional for all i. Using the fact that A has all nonzero offdiagonal entries, for $n > 2$ this proportionality can only occur when the positive vectors x and y in $\mathbf{C}^n$ are proportional. This, however, implies that

$$f_i(A) = r_i^x(A) = r_i^y(A) = g_i(A), \qquad i = 1, 2, \ldots, n, \tag{4.15}$$

which contradicts the assumption that f and g differ on A. Thus, for the case $\alpha = \beta$ and distinct minimal continuous f and g, the (α, β)-convolution of Eq. (4.8), i.e., the α-convolution of f and g, is not minimal continuous.

We consider now the case when $n > 2$ and $\alpha \neq \beta$ with $0 < \alpha < 1$, $0 \leqslant \beta \leqslant 1$. If h were minimal continuous, we would again necessarily have that equality holds throughout for all i in Eq. (4.11) in the application of Hölder's inequality, for each $A \in \mathbf{C}^{n,n}$ with nonzero offdiagonal entries. Hence, the vectors

$$(|a_{i,j}|^{\beta/\alpha}x_j/x_i)_{\substack{j=1\\ j\neq i}}^{n} \quad \text{and} \quad (|a_{i,j}|^{(1-\beta)/(1-\alpha)}y_j/y_i)_{\substack{j=1\\ j\neq i}}^{n} \tag{4.16}$$

are proportional for all i. Because $\beta \neq \alpha$, these proportionalities imply that all the products $\prod_{i=1}^{n} |a_{i,\sigma i}|$, for any cyclic permutation σ on $\{1, 2, \ldots, n\}$, are equal. But, it is clear that there is an $A \in \mathbf{C}^{n,n}$ with nonzero offdiagonal entries for which these products are *not* all equal. Thus, when $\alpha \neq \beta$, the (α, β)-convolution of Eq. (4.8) is not minimal continuous.

Q.E.D.

We can now complete the proof of Theorem 2. We must show that, for $n > 2$, (i) implies (ii′). Suppose $\ln f$ is not extreme in $\mathscr{L}_n{}^c$; there exist $0 < \alpha < 1$, $g, h \in \mathscr{G}_n{}^c$, $g \neq h$ such that $\ln f = \alpha \ln g + (1 - \alpha)\ln h$. This means that $f = g^\alpha h^{1-\alpha} \in \mathscr{G}_n{}^c$. If both g and h are minimal in $\mathscr{G}_n{}^c$, Theorem 4 tells us that f is not minimal in $\mathscr{G}_n{}^c$; on the other hand, if either of g and h is not minimal in $\mathscr{G}_n{}^c$, clearly neither is f. Thus (i) implies (ii′).

As we have just seen, Theorem 4 gives us that (α, β)-convolutions, with $0 < \alpha, \beta < 1$, of distinct (if $\alpha = \beta$) minimal continuous G-generating families in $\mathscr{P}_n$ are not minimal. Quite the same negative result can be deduced for the new G-functions of Nowosad [6]. To describe Nowosad's result, let ϕ be any monotonic norm on $\mathbf{C}^{n-1}$, and let Ψ be its conjugate (or polar) norm, i.e., for $x = (x_2, \ldots, x_n)$ and $y = (y_2, \ldots, y_n)$ in $\mathbf{C}^{n-1}$,

$$\Psi(x) \equiv \sup_{\phi(y)=1} \left|\sum x_i y_i\right|.$$

For these norms, one has the generalized Hölder inequality (cf. [4, p. 43])

$$\left|\sum_{i=2}^{n} x_i y_i\right| \leqslant \phi(x) \cdot \Psi(y). \tag{4.17}$$

Using the notation again that $A^\alpha = (|a_{i,j}|^\alpha)$ if $A = (a_{i,j}) \in \mathbf{C}^{n,n}$ for $0 \leqslant \alpha \leqslant 1$, we next let $(A)_i$ denote the ith row of A with its diagonal entry deleted; thus, $(A)_i$ can be regarded as a vector in $\mathbf{C}^{n-1}$. Then, Nowosad [6] proved that f defined by

$$f_i(A) = \phi[(A^\alpha)_i] \cdot \Psi[(A^T)_i^{1-\alpha}], \qquad i = 1, 2, \ldots, n, \tag{4.18}$$

is in $\mathscr{G}_n{}^c$ for any α with $0 \leqslant \alpha \leqslant 1$. We shall give a proof of this, and show, for any $0 < \alpha < 1$, that f is *not* minimal continuous for $n > 2$.

For A irreducible in $\mathbf{C}^{n,n}$, it can be shown (cf. [6, Lemma 4.3]) for each α with $0 \leqslant \alpha \leqslant 1$, that there is a positive diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n) \in \mathbf{C}^{n,n}$ such that

$$\phi[(A^\alpha)_i] = \phi\{[D^{-1}(A^\alpha)^T D]_i\}, \qquad i = 1, 2, \ldots, n, \tag{4.19}$$

or equivalently,

$$\phi[(DA^\alpha)_i] = \phi\{[(A^\alpha)^T D]_i\}, \qquad i = 1, 2, \ldots, n. \tag{4.19'}$$

For the vector $d \in \mathbf{C}^n$, $d > 0$, consider the minimal continuous G-function c^d defined by Eq. (2.4). We can write this, from Eq. (4.17), as

$$c_i{}^d(A) = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{j,i}| d_j/d_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \left\{\frac{|a_{j,i}|^\alpha d_j}{d_i}\right\} \{|a_{j,i}|^{1-\alpha}\}$$

$$\leqslant \phi\{[D^{-1}(A^T)^\alpha D]_i\} \Psi((A^T)_i^{1-\alpha}), \qquad i = 1, 2, \ldots, n.$$

Thus, from Eqs. (4.18) and (4.19),

$$c_i{}^d(A) \leqslant \phi[(A^\alpha)_i] \cdot \Psi[(A^T)_i^{1-\alpha}] = f_i(A), \qquad i = 1, 2, \ldots, n, \tag{4.20}$$

from which it is clear that f, as defined in Eq. (4.18), is in $\mathscr{G}_n{}^c$.

Our interest once again is in showing a negative analogue of Theorem 4 for the G-function of Eq. (4.18).

THEOREM 5. *For $n > 2$, no $f \in \mathscr{G}_n{}^c$, defined by Eq.* (4.18) *with $0 < \alpha < 1$, is minimal in $\mathscr{G}_n{}^c$.*

Proof. For $n > 2$, choose any positive vectors $x = (x_2, \ldots, x_n)$ and $y = (y_2, \ldots, y_n)$ in $\mathbf{C}^{n-1}$ which are not *dual vectors*, i.e., inequality holds in Eq. (4.17). Because Eq. (4.19′) is homogeneous in $D = \mathrm{diag}(d_1, \ldots, d_n)$, we can set $d_1 = 1$. Now, for $0 < \alpha < 1$, set

$$|a_{j,i}|^{\alpha} d_j = x_j, \qquad |a_{j,1}|^{1-\alpha} = y_j, \qquad |a_{1,j}|^{\alpha} = x_j, \qquad j = 2, \ldots, n. \tag{4.21}$$

These equations determine positive $d_2, \ldots, d_n$, and nonzero offdiagonal entries $|a_{1,j}|$ and $|a_{j,1}|$, $j = 2, \ldots, n$. Then, simply set all remaining entries of the matrix $A = (a_{i,j}) \in \mathbf{C}^{n,n}$ to zero. Because the nondiagonal entries in the first row and column of A are nonzero, then A is irreducible. Next, because we are using the non-dual vectors x and y in $\mathbf{C}^{n-1}$, then by construction, inequality holds for $i = 1$ in Eq. (4.20), while all the equations of Eq. (4.19′) are valid. In other words, since $c_1{}^d(A) < f_1(A)$, then $c^d \leqslant f$ with $c^d \neq f$. Thus, f is not minimal in $\mathscr{G}_n{}^c$. Q.E.D.

5. MINIMAL G-FUNCTIONS

To characterize minimal (not necessarily continuous) G-functions, we first need some auxiliary results from elementary graph theory. Given any reducible $A \in \mathbf{C}^{n,n}$, it is well-known (cf. [10, p. 46]) that there is a permutation matrix $P \in \mathbf{C}^{n,n}$, and a positive integer m, $2 \leqslant m \leqslant n$, such that

$$PAP^T = \begin{bmatrix} \tilde{A}_{1,1} & \tilde{A}_{1,2} & \cdots & \tilde{A}_{1,m} \\ 0 & \tilde{A}_{2,2} & \cdots & \tilde{A}_{2,m} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{A}_{m,m} \end{bmatrix}, \tag{5.1}$$

where each square submatrix $\tilde{A}_{k,k}$, $k = 1, 2, \ldots, m$, is either irreducible, or a 1×1 null matrix. The form Eq. (5.1) gives rise to a partitioning of $\{1, 2, \ldots, n\}$ into m disjoint nonempty sets $S_k = S_k(A)$, corresponding to the distinct connected components of the directed graph for A. The subsets S_k do not depend on the choice of the permutation matrix P which is used to obtain the from Eq. (5.1). For $i = 1, 2, \ldots, n$, let $\langle i \rangle$ denote the subset S_k containing i, and let $|\langle i \rangle|$ denote the cardinality of $\langle i \rangle$, i.e., the number of distinct elements in $\langle i \rangle$. Next, we define, from Eq. (5.1),

$$A_R = P^T \mathrm{diag}(\tilde{A}_{1,1}, \ldots, \tilde{A}_{m,m}) P; \tag{5.2}$$

in essence, A_R is obtained by setting to zero all offdiagonal blocks in the block-triangular matrix of Eq. (5.1). If $A \in \mathbf{C}^{n,n}$ is irreducible, we define $A_R = A$ and $\langle i \rangle = \{1, 2, \ldots, n\}$ for all $i = 1, 2, \ldots, n$.

We remark that if $A \in \mathbf{C}^{n,n}$ is reducible, it follows from Eq. (5.1) that, for any $f \in \mathscr{P}_n$,

$$P\mathscr{M}^f(A_R)P^T \equiv \operatorname{diag}[\mathscr{M}^f_{1,1}(A),\ldots,\mathscr{M}^f_{m,m}(A)], \tag{5.3}$$

where each $\mathscr{M}^f_{k,k}(A)$ is either a square irreducible matrix with nonpositive offdiagonal elements, or a 1×1 null matrix. If $A \in \mathbf{C}^{n,n}$ is irreducible, we define, in analogy with Eq. (5.1), $m = 1$, and set

$$\mathscr{M}^f(A) = \mathscr{M}^f_{1,1}(A). \tag{5.3'}$$

If $f \in \mathscr{G}_n$, then, by Proposition 1, $\mathscr{M}^f(A)$, $P\mathscr{M}^f(A)P^T$, and all the $\mathscr{M}^f_{k,k}(A)$ are M-matrices.

We now define, for each $x \in \mathbf{C}^n$ with $x > 0$,

$$\hat{r}_i^x(A) = \frac{1}{x_i}\sum_{\substack{j\in\langle i\rangle\\ j\neq i}} |a_{i,j}|x_j, \quad \hat{c}_i^x(A) = \frac{1}{x_i}\sum_{\substack{j\in\langle i\rangle\\ j\neq i}} |a_{j,i}|x_j, \qquad i = 1,2,\ldots,n \tag{5.4}$$

(we take $\hat{r}_i^x(A) = \hat{c}_i^x(A) = 0$ if $\langle i\rangle = \{i\}$). It is easy to see that $\hat{r}^x = (\hat{r}_1^x,\ldots,\hat{r}_n^x)$ and $\hat{c}^x = (\hat{c}_1^x,\ldots,\hat{c}_n^x)$ are in $\mathscr{G}_n$, and that $\hat{r}^x \lneqq r^x$ and $\hat{c}^x \lneqq c^x$. The functions $\hat{r}_i^x$ and $\hat{c}_i^x$ are, however, *not continuous*; if $e = (1, 1)^T$ and

$$A_t = \begin{bmatrix} 1 & 1 \\ t & 1 \end{bmatrix},$$

we have $\hat{r}_1^e(A_t) = \hat{c}_2^e(A_t) = 1$ for all $t > 0$, and yet $\hat{r}_1^e(A_0) = \hat{c}_2^e(A_0) = 0$.

Let $g \in \mathscr{P}_n$, such that, for every $A \in \mathbf{C}^{n,n}$,

$$g_i(A) > 0 \quad \text{whenever} \quad |\langle i\rangle| > 1. \tag{5.5}$$

We define $\hat{r}^g = (\hat{r}_1^g,\ldots,\hat{r}_n^g) \in \mathscr{P}_n$ by

$$\hat{r}_i(A) = \frac{1}{g_i(A)}\sum_{\substack{j\in\langle i\rangle\\ j\neq i}} |a_{i,j}|g_j(A), \qquad i = 1,2,\ldots,n, \tag{5.6}$$

(and $\hat{r}_i^g(A) = 0$ if $|\langle i\rangle| = 1$), for every $A \in \mathbf{C}^{n,n}$.

We can now characterize the minimal elements of $\mathscr{G}_n$.

THEOREM 6. *Let $f \in \mathscr{G}_n$. Then the following are equivalent:*

(i) *f is minimal in $\mathscr{G}_n$;*

(ii) *f is an extreme point of $\mathscr{G}_n$;*

(iii) *for every $A \in \mathbf{C}^{n,n}$, $\mathscr{M}^f_{k,k}(A)$ [cf. Eq. (5.3)] is a singular M-matrix for each $k = 1, 2,\ldots, m$;*

(iv) *for every $A \in \mathbf{C}^{n,n}$, there exists an $x \in \mathbf{C}^n$ with $x > 0$ (depending on A) such that*

$$f_i(A) = \hat{r}_i^x(A), \qquad i = 1, 2, \ldots, n; \tag{5.7}$$

(iv′) *there exists a* $g \in \mathscr{P}_n$ *such that, for every* $A \in \mathbf{C}^{n,n}$, *Eq.* (5.5) *holds, and*

$$f_i(A) = \hat{r}_i^g(A), \qquad i = 1, 2, \ldots, n. \tag{5.8}$$

Proof. We have already shown that (i) and (ii) are equivalent. We first show that (i) is equivalent with (iii) by contraposition. Suppose (i) does not hold. Then, there exists $g \in \mathscr{G}_n$ with $g \leqslant f$, $g \neq f$. Consequently, there exist an $A \in \mathbf{C}^{n,n}$ and an integer j, $1 \leqslant j \leqslant n$, for which Eq. (4.6) holds. Suppose $j \in S_k$. From Proposition 1, $\mathscr{M}^g_{k,k}(A)$ is an irreducible (or 1×1 null) M-matrix. Using Eq. (4.6), as in the proof of Theorem 2, we see that $\mathscr{M}^f_{k,k}(A)$ is nonsingular, and (iii) does not hold, i.e., (iii) implies (i).

For any $f \in \mathscr{G}_n$, we construct a $g \in \mathscr{G}_n$, for which $g \leqslant f$, by defining, for each $A \in \mathbf{C}^{n,n}$,

$$g_i(A) = f_i(A) - \lambda_k(A), \qquad i = 1, 2, \ldots, n, \tag{5.9}$$

where $i \in S_k$ and $\lambda_k(A)$ is the minimal nonnegative real eigenvalue of the M-matrix $\mathscr{M}^f_{k,k}(A)$. Now, suppose that (iii) does not hold. Then, there exists an $A \in \mathbf{C}^{n,n}$ for which some $\mathscr{M}^f_{k,k}(A)$ is nonsingular. Consequently, $g \neq f$, and (i) is violated, i.e., (i) implies (iii), and thus, (i) is equivalent with (iii).

We next show that (iii) is equivalent with (ii). Assume (iii); then $\mathscr{M}^f_{k,k}(A)$ is either an irreducible singular M-matrix, or a 1×1 null matrix for each $k = 1, 2, \ldots, m$. Thus, if $|S_k|$ denotes the number of elements in S_k, there exists a $x^{(k)}$ with $|S_k|$ positive components such that $M^f_{k,k}(A)x^{(k)} = 0$, i.e.,

$$f_i(A)x_i = \sum_{\substack{j \in S_k \\ j \neq i}} |a_{i,j}|x_j, \qquad i \in S_k. \tag{5.10}$$

The components of the $x^{(k)}$ form a vector $x \in \mathbf{C}^n$ with $x > 0$, for which Eq. (5.7) holds; i.e., (iii) implies (iv).

Assume next that (iv) holds. For each k, $k = 1, 2, \ldots, m$, let $X^{(k)}$ be an $|S_k| \times |S_k|$ positive diagonal matrix with diagonal entries x_i, $i \in S_k$. If $e^{(k)}$ is the $|S_k|$-tuple whose components are all unity, then Eq. (5.7) implies that $\mathscr{M}^f_{k,k}(A)X^{(k)}e^{(k)} = 0$; hence $\mathscr{M}^f_{k,k}(A)X^{(k)}$ and $\mathscr{M}^f_{k,k}(A)$ are singular. As this holds for all k, $k = 1, 2, \ldots, m$, then (iv) implies (iii).

That (iv) and (iv′) are equivalent is clear. Q.E.D.

COROLLARY 1. *Let f be minimal in $\mathscr{G}_n$, and assume that each f_i is monotone, i.e., if $A, B \in \mathbf{C}^{n,n}$ and $|a_{i,j}| \leqslant |b_{i,j}|$ for all $i \neq j$, $i, j = 1, 2, \ldots, n$, then $f_i(A) \leqslant f_i(B)$ for all $i = 1, 2, \ldots, n$. Then $f_i(A) = f_i(A_R)$ for all $i = 1, 2, \ldots, n$, and all $A \in \mathbf{C}^{n,n}$.*

COROLLARY 2. *Let $f \in \mathscr{G}_n$. Then for each $A \in \mathbf{C}^{n,n}$, there exists a vector $x \in \mathbf{C}^n$ with $x > 0$ for which*

$$f_i(A) \geqslant \hat{r}_i^x(A), \qquad i = 1, 2, \ldots, n. \tag{5.11}$$

Proof. For the G-function $f \in \mathscr{G}_n$, let $g \in \mathscr{G}_n$ be defined by Eq. (5.9). Because the matrices $\mathscr{M}_{k,k}^g(A)$ are by construction singular for all $k = 1, 2, \ldots, m$ and all $A \in \mathbf{C}^{n,n}$, then g is from Theorem 6 minimal in $\mathscr{G}_n$, with $g \leqslant f$. But as $g_i(A) = \hat{r}_i^x(A)$ from (iv) of Theorem 6, and $g \leqslant f$, then Eq. (5.11) follows. Q.E.D.

COROLLARY 3. *Let $g \in \mathscr{G}_n$. Then Eq. (5.5) holds for every $A \in \mathbf{C}^{n,n}$. If g is minimal in $\mathscr{G}_n$, then also, for every $A \in \mathbf{C}^{n,n}$, $g_i(A) = 0$ whenever $|\langle i \rangle| = 1$.*

COROLLARY 4. *For any $x \in \mathbf{C}^n$* with *$x > 0$, $\hat{r}^x$ and $\hat{c}^x$ are minimal G-functions.*

6. GENERALIZATION TO THE EXTENDED REAL NUMBERS

In Theorem 2, we have shown that, if f is minimal in $\mathscr{G}_n^c$, then there exists a $g \in \mathscr{I}_n^c$ such that $f(A) = f^g(A)$ for every irreducible $A \in \mathbf{C}^{n,n}$. We cannot, however, given an arbitrary $g \in \mathscr{I}_n^c$, always find a continuous extension of r^g to *all* of $\mathbf{C}^{n,n}$; for an example, let $n = 2$, and define, on irreducible $A \in \mathbf{C}^{2,2}$,

$$g(A) = (|a_{1,2}|^2, |a_{2,1}|^2).$$

Then $g \in \mathscr{I}_2^c$, and

$$r^g(A) = \left(\frac{|a_{2,1}|^2}{|a_{1,2}|}, \frac{|a_{1,2}|^2}{|a_{2,1}|} \right), \tag{6.1}$$

which clearly has no continuous extension to all of $\mathbf{C}^{2,2}$. If $g \in \mathscr{I}_n^c$ has a continuous extension $\tilde{g} \in \mathscr{P}_n^c$ for which $\tilde{g}_i(A) > 0$, all $i = 1, 2, \ldots, n$, all $A \in \mathbf{C}^{n,n}$, then r^g has a continuous extension $r^{\tilde{g}}$, defined by (4.1) for all

$A \in \mathbf{C}^{n,n}$. The question of finding (interesting and useful) necessary and sufficient conditions on $g \in \mathscr{I}_n{}^c$ so that such a continuous extension exists remains however open.

One could consider G-functions in the following extended setting. Let $\overline{\mathscr{P}}_n$, $n \geqslant 2$, be the collection of functions $f = (f_1, f_2, \ldots, f_n)$ such that for each $i = 1, 2, \ldots, n$, $+\infty \geqslant f_i(A) \geqslant 0$ for every $A \in \mathbf{C}^{n,n}$, and f_i depends only on moduli of offdiagonal entries. Then $f \in \overline{\mathscr{P}}_n$ is an *extended G-function* if, whenever $f(A)$ is finite, Eq. (2.1) implies that A is nonsingular. (Note that when any $f_i(A)$ is not finite, the union of the n disks Δ_k [cf. Eq. (2.2)] is the entire complex plane $\mathbf{C}$.) In this setting, we could extend r^g, defined in Eq. (6.1), to all of $\mathbf{C}^{2,2}$ in a natural way, and obtain an extended *G-function.*

We shall say that $f \in \overline{\mathscr{P}}_n$ is *continuous* if f is continuous (and hence finite) at every irreducible $A \in \mathbf{C}^{n,n}$, and if, for every reducible $A \in \mathbf{C}^{n,n}$,

$$f_i(A) = \varlimsup_{\substack{B \to A \\ B \text{ irreducible}}} f_i(B), \qquad i = 1, 2, \ldots, n \tag{6.2}$$

(i.e., f is actually upper semicontinuous at every reducible $A \in \mathbf{C}^{n,n}$). With this definition, Proposition 2 still holds (i.e., if A is reducible, $f(A)$ is finite, and $|a_{ii}| > f_i(A)$, $i = 1, 2, \ldots, n$, then A is nonsingular).

Now if $g \in \mathscr{I}_n{}^c$, $r^g \in \mathscr{I}_n{}^c$, we can define a continuous extension f of r^g using Eq. (6.2) for all reducible $A \in \mathbf{C}^{n,n}$. Then f is a continuous extended G-function, which is minimal among such functions. Conversely, if f is a minimal continuous extended G-function, then there exists a $g \in \mathscr{I}_n{}^c$ such that $f = r^g$ on irreducible $A \in \mathbf{C}^{n,n}$.

One final remark. If $x = x^{(0)} \in \mathbf{C}^n$ with $x > 0$, then, for every $A \in \mathbf{C}^{n,n}$ which has a nonzero offdiagonal entry in each row, we can define

$$r^x_{(m)}(A) = r^{[r^x_{(m-1)}(A)]}(A), \qquad m = 1, 2, \ldots. \tag{6.3}$$

If also A is irreducible and primitive (cf. [10, p. 35]), it is not difficult to see (cf. [10, p. 44]) that

$$\lim_{m \to \infty} r^x_{(m)i}(A) = \rho(A), \qquad i = 1, 2, \ldots, m, \tag{6.4}$$

where $\rho(A)$ is the maximal eigenvalue of the nonnegative matrix $\mathscr{Q}(A) = (|a_{i,j}| - \delta_{i,j}|a_{i,i}|)$. This "limiting function," as we have seen in Sec. 4, does extend to a continuous G-function on $\mathbf{C}^{n,n}$.

7. MINIMAL G-FUNCTIONS WITH SMALL DOMAINS OF DEPENDENCE

For any $f = (f_1, \ldots, f_n) \in \mathscr{P}_n$, we say [3] that f_k *depends* on the ordered pair of positive integers (i, j) where $i \neq j$ and $1 \leqslant i, j \leqslant n$, if there exist $A = (a_{k,l}) \in \mathbf{C}^{n,n}$ and $B = (b_{k,l}) \in \mathbf{C}^{n,n}$ such that $|a_{k,l}| = |b_{k,l}|$ for all $k \neq l$ with $(k, l) \neq (i, j)$, for which $f_k(A) \neq f_k(B)$. We then define

$$D(f_k) = \{(i, j)\colon \ 1 \leqslant i, j \leqslant n \quad \text{and } f_k \text{ depends on } (i, j)\} \tag{7.1}$$

as the *domain of dependence* of f_k. With this notation, we now prove the following proposition.[2]

PROPOSITION 3. *Let $f \in \mathscr{G}_n$. For each ordered pair (i, j) with $1 \leqslant i, j \leqslant n$ and $i \neq j$, there exists a positive integer k with $1 \leqslant k \leqslant n$ such that $(i, j) \in D(f_k)$.*

Proof. We consider the ordered pair (i, j) with $1 \leqslant i, j \leqslant n$ and $i \neq j$. Suppose $(i, j) \notin D(f_k)$ for all $k = 1, 2, \ldots, n$. Then, for any $A \in \mathbf{C}^{n,n}$, each $f_k(A)$ is *independent* of $a_{i,j}$. It would then be possible to find a nonsingular irreducible M-matrix $B = (b_{k,l}) \in \mathbf{R}^{n,n}$ with

$$|b_{l,l}| > f_l(B) \quad \text{for all} \quad l = 1, 2, \ldots, n.$$

Without affecting these inequalities, we could *decrease* the element $-|b_{i,j}|$ until B becomes singular, contradicting that f is a G-function. Q.E.D.

Most well-known G-functions (cf. [2]) have

$$D(f_k) = \{(k, l)\colon \ 1 \leqslant l \leqslant n, \quad l \neq k\} \tag{7.2}$$

for $k = 1, 2, \ldots, n$, or

$$D(f_k) = \{(l, k)\colon \ 1 \leqslant l \leqslant n, \quad l \neq k\} \tag{7.3}$$

for $k = 1, 2, \ldots, n$, or are obtained by convolution from functions in $\mathscr{P}_n$ with such domains of dependence. Among these, the row and column sums r^x and c^x have always played a central position in Gerschgorin-type arguments for matrices. The next surprising result gives yet another reason for this.

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[2] After submitting this paper, we learned that Proposition 3 was independently included in a talk by A. J. Hoffman at a Conference on Graph Theory at St. John University in the summer of 1970.

THEOREM 7. *Let f be minimal in $\mathcal{G}_n{}^c$. If Eq. (7.2) is valid for one particular k, and if*

$$\left\{\bigcup_{\substack{l=1\\l\neq k}}^{n} D(f_l)\right\} \quad \textit{and} \quad D(f_k) \textit{ are disjoint,} \tag{7.4}$$

then Eq. (7.2) is valid for all k, $k = 1, 2,\ldots, n$, and there exists an $x \in \mathbf{C}^n$ with $x > 0$ (independent of A) such that $f = r^x$. Similarly, if Eq. (7.3) is valid for one particular k, and if Eq. (7.4) is valid, then Eq. (7.3) is valid for all k, $k = 1, 2,\ldots, n$, and there exists an $x \in \mathbf{C}^n$ with $x > 0$ (independent of A) such that $f = c^x$.

Proof. Assume that f is minimal in $\mathcal{G}_n{}^c$, that Eq. (7.2) is valid for $k = 1$, and that Eq. (7.4) holds for $k = 1$. Then, from Theorem 2, $\mathcal{M}^f(A)$ is a singular M-matrix for each $A \in \mathbf{C}^{n,n}$. For any irreducible $A = (a_{i,j}) \in \mathbf{C}^{n,n}$ with all nonzero offdiagonal entries, there is a unique normalized $x = (1, x_2,\ldots, x_n)^T$ in $\mathbf{C}^n$ with $x > 0$ such that $\mathcal{M}^f(A)x = 0$, and thus

$$f_k(A) = \frac{1}{x_k}\sum_{\substack{j=1\\j\neq k}}^{n} |a_{k,j}|x_j \quad \text{for all} \quad k = 1, 2,\ldots, n.$$

Let $B \in \mathbf{C}^{n-1,n-1}$ be the principal submatrix of $\mathcal{M}^f(A)$ obtained by deleting the first row and first column from $\mathcal{M}^f(A)$. It is easily seen (cf. [10, p. 30]) that B must be nonsingular. Since $\mathcal{M}^f(A)x = 0$ can be written as the pair of equations

$$B\cdot(x_2,\ldots, x_n)^T - (|a_{2,1}|,\ldots, |a_{n,1}|)^T = 0; \quad f_1(A) - \sum_{j=2}^{n} |a_{1,j}|x_j = 0, \tag{7.5}$$

then as B is nonsingular, the vector $(x_2,\ldots, x_n)^T \in \mathbf{C}^{n-1}$ can be expressed simply as

$$(x_2,\ldots, x_n)^T = B^{-1}(|a_{2,1}|,\ldots, |a_{n,1}|)^T. \tag{7.6}$$

But from Eq. (7.4), the $f_j(A)$ for $j = 2,\ldots, n$, and hence B^{-1}, are all independent of $|a_{1,2}|,\ldots, |a_{1,n}|$. Thus, from Eq. (7.6), the components of $x = (1, x_2,\ldots, x_n)^T$ are independent of $|a_{1,2}|,\ldots, |a_{1,n}|$. This means that if we now vary the matrix A only in the components $|a_{1,2}|,\ldots, |a_{1,n}|$, keeping A irreducible, the second equation of (7.5) remains valid where $x_2,\ldots, x_n$ are fixed, i.e.,

$$f_1(A) = \sum_{j=2}^{n} |a_{1,j}|x_j = r_1{}^x(A).$$

Hence, from the continuity of f_1, the above expression must be valid for all $A \in \mathbf{C}^{n,n}$.

Next, suppose that $x_2(A), \ldots, x_n(A)$ can vary when A varies over the matrices for which $(a_{ij}) \neq 0$, $i = 2, \ldots, n$, $j = 1, \ldots, n$, $i \neq j$, and for which the first row of A has the explicit form $e_k = (0, 0, \ldots, 1, 0, \ldots, 0)$, $k = 2, \ldots, n$. Because $f_1(A) = r_1{}^x(A)$ from the second equation of (7.5), we have in this case that $f_1(A) = x_k(A)$. On the other hand, since f_1 can, by hypothesis, depend only on $\{(1, l)\colon\ 2 \leqslant l \leqslant n\}$, while x_k, from Eq. (7.6), is independent of the first row of A, then $f_1(A) = x_k$. Hence $x_2, \ldots, x_n$ from Eq. (7.6) are fixed for *all* $A \in \mathbf{C}^{n,n}$. It thus follows that Eq. (7.2) is valid for *all* $k = 1, 2, \ldots, n$, and $f = r^x$, where x is independent of A. The proof of the rest of the theorem is similar. Q.E.D.

In the rest of this section, we make specific use of the main result of Hoffman and Varga [3], which we state below as Theorem 8. For notation, we say that $f \in \mathscr{P}_n$ is *homogeneous* (of degree unity) if, for every $\lambda > 0$ and every $A \in \mathbf{C}^{n,n}$,

$$f_k(\lambda A) = \lambda f_k(A), \qquad k = 1, 2, \ldots, n, \tag{7.7}$$

and we say that f is *bounded on bounded sets* if, for all $A = (a_{i,j}) \in \mathbf{C}^{n,n}$ with $|a_{i,j}| \leqslant c$ for all $i, j = 1, 2, \ldots, n$ with $i \neq j$, there exist positive constants $M_k(c)$ such that

$$f_k(A) \leqslant M_k(c), \qquad k = 1, 2, \ldots, n. \tag{7.8}$$

THEOREM 8. *Let $D_1, D_2, \ldots, D_n$ be subsets of the set of all ordered pairs of positive integers (i, j), with $1 \leqslant i, j \leqslant n$, and $i \neq j$. Then, there exists an f in $\mathscr{G}_n$, with f homogeneous and bounded on bounded sets, satisfying*

$$D_k = D(f_k), \qquad k = 1, 2, \ldots, n, \tag{7.9}$$

if and only if, for every subset $S \subset \{1, 2, \ldots, n\}$ with $|S| \geqslant 2$, for every cyclic permutation σ of S and for every nonempty subset $T \subset S$,

$$|\{i\colon\ i \in S \quad \text{and} \quad \{(i, \sigma i)\} \in \bigcup_{k \in T} D_k\}| \geqslant |T|. \tag{7.10}$$

With Theorem 8, we establish

PROPOSITION 4. *Let $f \in \mathscr{G}_n$ with f homogeneous and bounded on bounded sets. Then*

$$|D(f_k)| \geqslant n - 1 \quad \textit{for} \quad k = 1, 2, \ldots, n. \tag{7.11}$$

Proof. Let k be a fixed positive integer with $1 \leqslant k \leqslant n$. If $S_j \equiv \{k, j\}$ for any j with $1 \leqslant j \leqslant n$ and $j \neq k$, then $|S_j| = 2$. Next, let $T \equiv \{k\} \subset S_j$. Applying Eq. (7.10) of Theorem 8, it is clear that either (k, j) or (j, k) is in $D(f_k)$. Thus, letting j run from 1 to n, $j \neq k$, $D(f_k)$ must contain at least $n - 1$ distinct ordered pairs, i.e., $|D(f_k)| \geqslant n - 1$, which establishes Eq. (7.11). Q.E.D.

We note that, as we saw at the end of Sec. 4, there are non-homogeneous continuous G-functions for which (7.11) does *not* hold.

THEOREM 9. *Let $f \in \mathscr{G}_n$ for which f is homogeneous and bounded on bounded sets, and for which*

$$|D(f_k)| = n - 1 \quad \textit{for} \quad k = 1, 2, \ldots, n. \tag{7.12}$$

If $(1, 2) \in D(f_1)$, then Eq. (7.2) holds for all $k = 1, 2, \ldots, n$. Otherwise, $(2, 1) \in D(f_1)$, and Eq. (7.3) holds for all $k = 1, 2, \ldots, n$.

Proof. From Proposition 3 and from the assumption of Eq. (7.12), it is clear that the $n(n - 1)$ ordered pairs of integers (i, j) with $i \neq j$ and $1 \leqslant i, j \leqslant n$, must be distributed among the $D(f_k)$'s in such a way so that the $D(f_k)$'s are *pairwise disjoint.*

First, consider the set $D(f_1)$. As in the proof of Proposition 4, if $S \equiv \{1, j\}$ for any $2 \leqslant j \leqslant n$ and if $T = \{1\} \subset S$, then Eq. (7.10) of Theorem 8 gives us that either $(1, j)$ or $(j, 1)$ is in $D(f_1)$ for each $j = 2, \ldots, n$. In particular, either (1, 2) or (2, 1) is in $D(f_1)$. *Assume* $(1, 2) \in D(f_1)$. Then, $(2, 1) \notin D(f_1)$; otherwise $|D(f_1)| > n - 1$. For $n > 2$, consider $S_l \equiv \{1, 2, l\}$ with $l = 3, \ldots, n$, and $T = \{1\} \subset S_l$. For the particular cyclic permutation σ of S_l defined by $\sigma 1 = l$, $\sigma 2 = 1$, $\sigma l = 2$, it follows from Eq. (7.10) that at least one of the pairs $(1, l)$, $(2, 1)$, and $(l, 2)$ is in $D(f_1)$, i.e., $(1, l)$ or $(l, 2)$ is in $D(f_1)$. Because $|D(f_1)| = n - 1$ and because $(1, j)$ or $(j, 1)$ must be in $D(f_1)$ for each $j = 2, \ldots, n$, then $(l, 2) \notin D(f_1)$ for any $l = 3, \ldots, n$, so that $(1, l) \in D(f_1)$ for each $l = 3, \ldots, n$. Thus, $D(f_1) = \{(1, j)\colon\ j = 2, 3, \ldots, n\}$, the special case $k = 1$ of Eq. (7.2). In a similar way, one establishes the general results of Eqs. (7.2) and (7.3). Q.E.D.

By the methods of [2], we can construct continuous, nonhomogeneous G-functions, satisfying Eq. (7.12), for which the conclusion of Theorem 9 does not hold.

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